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Persistence in systems with algebraic interaction

Iaroslav Ispolatov

Department of Physics, McGill University, 3600 rue University, Montréal, Québec, Canada H3A 2T8
and Chemistry Department, Baker Laboratory, Cornell University, Ithaca, New York 14853

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Persistence in coarsening one-dimensional spin systems with a power-law interaction $r^{-1-\sigma}$ is considered. Numerical studies indicate that for sufficiently large values of the interaction exponent σ ($\sigma \geqslant 1/2$ in our simulations), persistence decays as an algebraic function of the length scale L, $P(L) \sim L^{-\theta}$. The persistence exponent θ is found to be independent on the force exponent σ and close to its value for the extremal ($\sigma \to \infty$) model, $\overline{\theta} = 0.175\,075\,88\cdots$. For smaller values of the force exponent ($\sigma < 1/2$), finite size effects prevent the system from reaching the asymptotic regime. Scaling arguments suggest that in order to avoid significant boundary effects for small σ, the system size should grow as $[O(1/\sigma)]^{1/\sigma}$.
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Coarsening dynamics of one-dimensional (1D) systems with a power-law $V(r) \sim r^{-\sigma-1}$ interaction between spins has recently been studied by Lee and Cardy [1], and Rutenberg and Bray [2]. It had been established that after quenching from a high-temperature disordered phase to $T=0$ these systems develop a domain structure characterized by a single length scale $L(t)$. A naive argument based on the law of motion for domain walls, $\dot{L} \sim L^{-\sigma}$ (where $L^{-\sigma}$ is a typical force between domain walls), produces an asymptotically correct time dependence of L,

$$L(t) \sim t^{1/1+\sigma}. \tag{1}$$

Other properties of this system, including correlation functions and domain size distribution, have been studied in [2] as well.

In this paper we shall look at another facet of 1D phase-ordering systems with a power-law interaction; what fraction P of spins have never changed sign up to the time t? Or, equivalently, what fraction of the space has never been crossed by a domain wall? Such a property of coarsening systems is usually called persistence and has recently become a major subject of research in statistical physics [3–7]. Let us briefly review some known results in this field relevant to our problem. In [3] the exact solution was found for persistence in an ordering system described by the noiseless time-dependent Ginzburg-Landau equation. In the long time asymptotic regime this model can be viewed as an infinitely short-range $\sigma \to \infty$ limit of the system with power-law interaction. In this limit, coarsening proceeds by consecutive shrinking and disappearance of the current smallest domains in the system, while other domain boundaries remain virtually motionless. It was established in [3] that persistence at a stage of evolution when the average domain size L is proportional to $L^{-\overline{\theta}}$, where the exponent $\overline{\theta} = 0.175\,07\cdots$ is the solution of the implicit integral equation.

$$\int_0^\infty dx\, x^{-1-\overline{\theta}} \exp[-x],$$

$$[(1-x-\exp[-x])\exp[r(x)] + 2\overline{\theta}x + \overline{\theta}x^2 \exp[-r(x)]], \tag{2}$$

where $r(x) \equiv \int_x^\infty dy \exp[-y]/y + \ln[x]$.

In [4] persistence exponents have been calculated for coarsening 1D Potts models with Glauber dynamics. For the

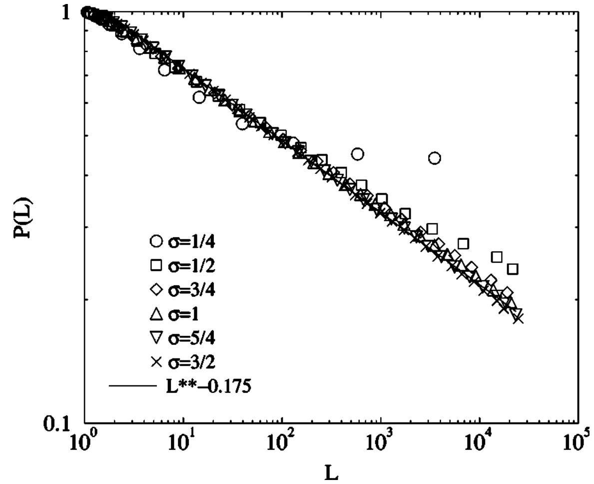

FIG. 1. Plot of persistence $P(L)$ vs average domain size L for various force exponents σ. The straight line corresponds to $P(L) \sim L^{-\bar{\theta}}$.

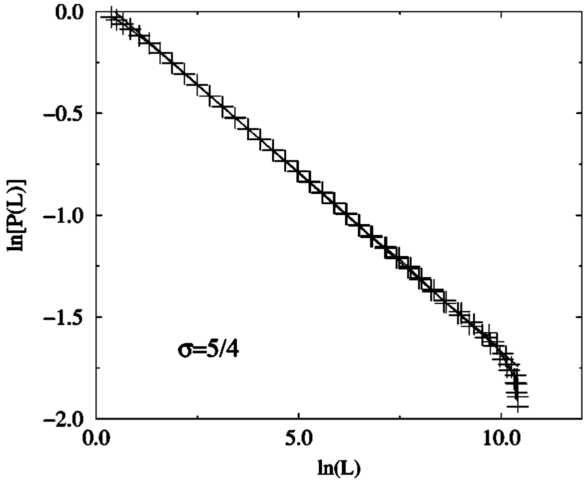

FIG. 2. Log-log plot of persistence $P(L)$ vs average domain size L force exponents $\sigma=5/4$ with statistical error bars. The straight line corresponds to $P(L) \sim L^{-\bar{\theta}}$.

two-state Potts (Ising) model, persistence decays as $t^{-\theta}$, $\theta = 3/8$, or in terms of the average domain size L, $P(L) \sim L^{-3/4}$.

The following conclusion can be drawn from a comparison of persistence exponents for extremal and Glauber dynamics. Extremal dynamics is more efficient in preserving persistence, since the motion of domain walls is always directed towards their ultimate annihilation partners while, in the case of Glauber dynamics, domain walls perform random walks and sweep through a larger amount of space, which otherwise could have remained persistent. The extremal dynamics exponent $\bar{\theta}$ sets a lower bound on persistence exponents for systems with a finite force exponent σ. It is easy to visualize a scenario when a domain wall first moves away from its ultimate annihilation partner, and then, after the stronger force source disappears, it turns back. Such events result in spin flips on parts of the line that belong to a surviving domain and would have been left untouched in the extremal dynamics case. The results presented below suggest that this lower boundary $\bar{\theta} = 0.175\,07\cdots$ is in fact the exact value of the persistent exponent for arbitrary $\sigma > 0$.

Let us formally introduce our model; we consider coarsening of the 1D two-state spin system with a long-range ferromagnetic Hamiltonian:

$$H = \frac{-4}{\sigma} \sum_{i>j} \frac{s_i s_j}{(x_i - x_j)^{\sigma+1}}. \tag{3}$$

After quenching from a high-temperature random phase to $T=0$, coarsening dynamics for this system is determined by the motion of domain walls, governed by the Langevin equation. The velocity of a wall is equal to the sum of pairwise forces from other walls, with walls of the same signs repelling and walls of the opposite signs attracting each other:

$$\frac{dr_i}{dt} = \sum_{j \neq i} (-1)^{i+j} \,\mathrm{sgn}(r_i - r_j) F_{ij}, \tag{4}$$

$$F_{ij} = \frac{1}{|r_i - r_j|^{\sigma}}. \tag{5}$$

When the adjacent walls meet, they annihilate. As we mentioned, the degree of coarsening is uniquely characterized by a typical domain size $L(t) \sim t^{1/(1+\sigma)}$. We measure the fraction of space $P(L)$ that has never been crossed by a single domain wall as a function of this length scale $L(t)$. We perform molecular dynamics simulations of the model for the following values of the force exponent: $\sigma = 3/2, 5/4, 1, 3/4, 1/2, 1/4$. Each run starts with a system consisting of $N_0 = 100\,000$ domain walls with exponential distribution of domain sizes, $W(L_0) = \exp(-L_0)$. Results for each σ are averaged over 20 initial configurations. Open boundary conditions with no replicas added to the boundaries are used. To speed up the evaluation of forces, a 1D multipole expansion has been performed, and terms of up to quadrupole order were taken into account [8].

The results for persistence as a function of the average domain length L are presented in log-log form in Fig. 1. Except for small force exponents ($\sigma = 1/4$ and later evolution stages for $\sigma = 1/2$), all of the curves collapse at a line with a slope ≈ -0.175, which corresponds to $\sigma = \infty$ extremal model. Statistical error bars are shown in Fig. 2 for a single set of data ($\sigma = 5/4$).

Our simulations suggests that scaling of persistence, corresponding to $\sigma = \infty$, is valid for all other not very small σ. The following asymptotic argument helps us to understand why this is so. At any current moment of time, persistent spins are mostly contained in the domains that were expanding at almost all previous stages of coarsening; i.e., these domains were larger than the average at those stages. If one of these large domains is surrounded by two small neighbors, it would most probably grow outwards, and no spin flips, in addition to those inevitably caused by directed coarsening itself, would happen. The situation may be different if two or three big domains are adjacent to each other; their domain walls may wander and get inside the territory of the future survivor, causing some excessive spin flips.

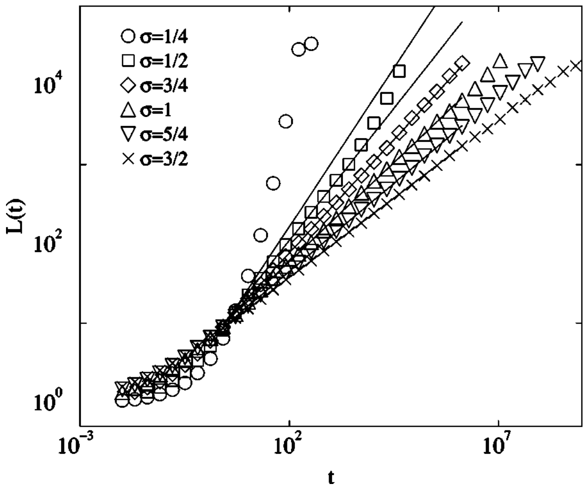

FIG. 3. Plot of average domain size $L(t)$ vs time t for various force exponents σ. Straight lines correspond to scaling predictions, $L(t)\sim t^{1/1+\sigma}$.

We can estimate the characteristic scale of such a persistence-losing event. A typical distance ΔL that a wall of large domain of size L_l, surrounded by a group of domains of similar sizes, travels during time t is

$$\Delta L \sim L_l-(L_l^{(1+\sigma)}-t)^{1/1+\sigma} \approx L(t)\left[\frac{L(t)}{L_l}\right]^{\sigma}. \qquad (6)$$

Here $L(t)\sim t^{1/1+\sigma}$ is the average domain size at time t. For positive σ, ΔL becomes small compared to $L(t)$ when $L(t)/L_l \ll 1$; hence the number of spin flips in addition to those present in extremal dynamics coarsening scenario becomes negligible. Another conclusion that follows from Eq. (6) is that for small σ, the crossover time to $P(L)\sim L^{-\bar{\theta}}$ scaling must be larger since the system must develop a structure that includes sufficiently large domains.

However, besides long initial transitional times, there is another reason for the breakdown of scaling for small σ that we observed in our simulations. Let us first consider the opposite of the $\sigma=\infty$ case of $\sigma=0$. In this limit forces are distance independent, and the domain wall dynamics (4) is described by the equation

$$\frac{dr_i}{dt}=\sum_{j\neq i}(-1)^{i+j}\,\mathrm{sgn}(r_i-r_j). \qquad (7)$$

If we consider a system with even number of domains where the domain walls come in pairs, the sum in Eq. (7) is equal to ± 1. That means that all walls have the same constant velocity with odd-number walls moving to the left and even-number moving to the right. The whole system becomes a collection of independently collapsing and growing domains. This clearly violates the scaling (1); in fact, the $\sigma=0$ system has two length scales, L_0-2vt and L_0+2vt, where L_0 is the average initial domain length and $v=1$ is the velocity of domain walls. For an exponential distribution of initial domain sizes, $W(L_0)=\exp(-L_0)$, persistence can be expressed as

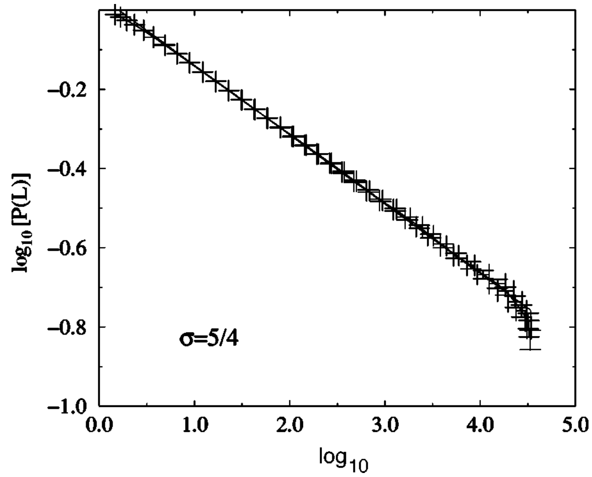

FIG. 4. Number of domain walls $B(L)$ that move opposite to the direction prescribed by boundary effects vs average domain length L for various force exponents σ.

$$\tilde{P}(t)=\frac{\int_0^\infty W(x)dx+\int_{2vt}^\infty W(x)dx}{2\int_0^\infty W(x)dx}=\frac{1+\exp[-2t]}{2}, \qquad (8)$$

with two terms in the numerator describing contributions from growing and shrinking domains. Systems with few particles and small σ coarsen almost according to the $\sigma=0$ scenario; particles across the whole system feel the presence of the boundary. Odd- and even-number walls tend to move predominantly to the left and right, respectively, independent of the position of their nearest neighbors.

To probe whether the deviation from $P(L)\sim L^{-\bar{\theta}}$ scaling in persistence behavior is caused by $\sigma=0$ finite size effects, we do the following measurements. First, for the system of the same initial size ($N=10^5$) we plot the average domain length $L(t)$ as a function of time and compare it to the $L\sim t^{1/1+\sigma}$ prediction.

Results for this simulation are presented in Fig. 3. One can see that the system with $\sigma=1/4$ is never in scaling regime (1), and the system with $\sigma=1/2$ behaves according to Eq. (1) only up to some intermediate stage of evolution. For all other force exponents $\sigma>1/2$, for a certain period of evolution after a short transitional time, typical domain sizes scale according to Eq. (1).

We also perform a direct check of whether the system feels the presence of the boundaries, i.e., we count the fraction of domain walls $B(L)$ that move opposite to the direction prescribed by the boundary effects. In Fig. 4 we plot the fraction of even-number domain walls moving to the right and odd-number domain walls moving to the left; initially the systems consist of the same number of domains, $N=10^5$. For finite $\sigma>0$ and a truly infinite system, this fraction should be equal to 1/2, for $\sigma=0$ it should be 0. We observe that, according to the $B(L)$ criteria, our system is never in the true infinite-size regime for $\sigma=1/4$, finite size effects are becoming evident for $\sigma=1/2$ even at early stages of evolution, and the boundary effects could be neglected only for $\sigma\geq 3/4$.

Comparing Figs. 1, 3, and 4, one can note that persistence $P(L)$ and the typical domain size $L(t)$ are less sensitive to the finite size effects than $B(L)$. When the significant fraction of the domain walls moves in the direction prescribed by the boundaries [$B(L)\approx 1/4$ for $\sigma=1/2$, $L=10^2$], $P(L)$ and $L(t)$ are still in the scaling regime. A possible explanation for the relative robustness of the behavior of the average domain size $P(L)$ and persistence $L(t)$ is that the main contribution to these quantities comes from the large domains, while for $B(L)$ we count the number of domain walls indiscriminate of the domain sizes.

Finally, we present a rough estimate of how big a system should be for a particular value of $\sigma\ll 1$ to avoid significant finite-size effects. We evaluate a typical ''local'' force F_{1-2}, exerted on a test domain wall by a dipole pair of neighboring domain walls,

$$F_{1-2}\approx\left(\frac{1}{L}\right)^{\sigma}-\left(\frac{1}{2L}\right)^{\sigma}=\left(\frac{1}{L}\right)^{\sigma}[\sigma\ln 2+\mathcal{O}(\sigma^2)], \quad (9)$$

and compare it to a ''boundary'' force F_N, exerted on the same test domain wall in the middle of the system, by a single domain wall near the edge of the system.

$$F_N\approx\left(\frac{2}{NL}\right)^{\sigma}. \quad (10)$$

Here L and N are the typical domain length and the number of domains in the system. The boundary effects become significant when these forces are of the same order. It follows that for $\sigma\rightarrow 0$, the minimum number of particles to avoid finite size effects N_{min} grows very fast:

$$N_{min}\sim\left(\frac{\mathcal{C}}{\sigma}\right)^{1/\sigma}. \quad (11)$$

In summary, we presented numerical evidence and a scaling argument suggesting the universality of persistent exponent for extremal model, $\overline{\theta}=0.175\,075\,88\cdots$, for models with arbitrary force exponents $\sigma>0$. We found that a deviation from scaling for persistence, which happens for small σ, is accompanied by a similar deviation from scaling for a typical domain size $L(t)$ and is caused by finite size effects that cause crossover to a $\sigma=0$ coarsening scenario. We estimated that in order to avoid boundary effects, the system size should grow as $[\mathcal{O}(1/\sigma)]^{1/\sigma}$. A possible extension of this work is for higher dimensional systems, though the duality between domain walls and spin dynamics that was extensively used for this work may not be so straightforward to apply.

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